

New Approach to N -body Relativistic Quantum Mechanics

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Abstract

In this paper, we propose a new approach to the relativistic quantum mechanics for many-body, which is a self-consistent system constructed by juxtaposed but mutually coupled nonlinear Dirac's equations. The classical approximation of this approach provides the exact Newtonian dynamics for many-body, and the nonrelativistic approximation gives the complete Schrödinger equation for many-body.

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1 Introduction

The nonrelativistic quantum mechanics for many body is described by the Schrödinger equation, which has successful and fruitful applications in quantum chemistry, quantum statistics, condensed matter physics, etc. The wonderful properties of this equation leads to many refined mathematical methods to solve the eigenvalues and eigenstates[1].

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But the corresponding relativistic theory is an unsettled problem. There are several approaches for this problem. Early in 1937, Fock proposed a 4+1 dimensional covariant Schrödinger-like equation with a scalar time[2]. In 1941, Stückelberg also provided a 4+1 dimensional covariant equation but without specific mass[3]. The equation was improved and generalized to the case with multi proper time[4, 5, 6, 7, 8, 9, 10].

A manifestly covariant approach derived from quantum field theory was introduced by Bethe and Salpeter[11], and developed in [12, 13]. But it is too complicated for practical calculation for more than 2 particles. A alternative version is the mass shell constraints approach given in [14, 15, 16, 17, 18], which has to treat multi relative times. This approach also is effective for two or three bodies problem. In [19, 20], After analyzing the former approaches, the authors proposed a new method to define a uniform time for n-particle with the same mass. They applied generalized Foldy-Wouthuysen transformation to the positive energy part of the Hamiltonian, and computed the charmonium and bottonium eigenstates.

Although the authors of [19] mentioned that to treat relativistic quantum mechanics for many body problem as a field theory was essentially abandoned. However all matters are 4-dimensional existence, whose intrinsic properties should be naturally described by 3+1 dimensional field variables. In this paper, we propose a new approach in the form of nonlinear coupled field theory, which assign a different spinor field ϕ_k instead of coordinate x_k^μ to each particle. In this version every thing can be well defined and conveniently for mathematical treatment. We derive the Newtonian mechanics for N -electron via clear definition and deduction, and derive the corresponding N -body Schrödinger equation under nonrelativistic approximation. The results show that it seems to be a meaningful model.

2 The Fundamental Equations

Denote the Minkowski metric by $\eta_{\mu\nu} = \text{diag}[1, -1, -1, -1]$, Pauli matrices by

$$\vec{\sigma} = (\sigma^j) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \quad (2.1)$$

Define 4×4 Hermitian matrices as follows

$$\alpha^\mu = \left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \right\}, \quad \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}. \quad (2.2)$$

For N electrons e_1, e_2, \dots, e_N moving in the external potential V^μ generated by external charge ρ^μ , we find that the following Lagrangian give a good description for the system of N -electron,

$$\mathcal{L} = \sum_{k=1}^n \phi_k^+ [\alpha^\mu (\hbar i \partial_\mu - e A_\mu) - \mu \gamma] \phi_k - \rho_\mu A^\mu + \frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + F, \quad (2.3)$$

where $\mu > 0$ is a constant mass, which takes one value for the same kind particles. F is the nonlinear coupling term, we take the following form as example for calculation

$$F = \frac{1}{2} w (Q_\mu Q^\mu - \sum_{k=1}^N \check{\beta}_k^2), \quad w > 0, \quad (2.4)$$

where Q^μ is the total current

$$Q^\mu = \sum_{k=1}^N q_k^\mu, \quad q_k^\mu = \phi_k^+ \alpha^\mu \phi_k, \quad \check{\beta}_k = \phi_k^+ \beta \phi_k, \quad (2.5)$$

In this paper, we adopt the Hermitian matrices (2.2) instead of Dirac matrices γ^μ , because this form is more convenient for calculation.

The variation of (2.3) with respect to ϕ_k gives the dynamic equation for e_k

$$\alpha^\mu (\hbar i \partial_\mu - e A_\mu + w Q_\mu) \phi_k = (\mu \gamma + w \check{\beta}_k \beta) \phi_k, \quad (2.6)$$

or in the Hamiltonian form as usual

$$\hbar i \partial_t \phi_k = \hat{H}_k \phi_k, \quad (k = 1, 2, \dots, N) \quad (2.7)$$

where the nonlinear Hamiltonian operator \hat{H}_k is defined by

$$\hat{H}_k \equiv \vec{\alpha} \cdot [-\hbar i \nabla - e \vec{A} + w(\vec{Q} - \vec{q}_k)] + e A_0 - w(Q_0 - q_{k0}) + (\mu - w \check{\gamma}_k) \gamma. \quad (2.8)$$

with $\check{\gamma}_k \equiv \phi_k^+ \gamma \phi_k$. In (2.8) we have used the Pauli-Fierz identity $q_{k\mu} q_k^\mu = \check{\beta}_k^2 + \check{\gamma}_k^2$ [21]. In this paper we denote $\vec{A} = (A^1, A^2, A^3)$ to be the spatial part of a contravariant vector A^μ .

For each bispinor ϕ_k , it is easy to check that the current conservation law holds $\partial_\mu q_k^\mu = 0$, so we can take the normalizing condition as

$$\int_{R^3} q_k^0 d^3x = \int_{R^3} |\phi_k|^2 d^3x = 1, \quad (\forall k). \quad (2.9)$$

The electromagnetic theory is the following Maxwell equation

$$\begin{cases} \partial_\mu q^\mu = \partial_\mu A^\mu = 0, & \partial^\alpha \partial_\alpha A^\mu = eQ^\mu + \rho^\mu, \\ \vec{E} = -\nabla A^0 - \partial_0 \vec{A}, & \vec{B} = \nabla \times \vec{A}, \\ \nabla \cdot \vec{E} = eQ^0 + \rho^0, & \nabla \times \vec{E} = -\partial_0 \vec{B}, \\ \nabla \cdot \vec{B} = 0, & \nabla \times \vec{B} = \partial_0 \vec{E} + e\vec{Q} + \vec{\rho}, \end{cases} \quad (2.10)$$

(2.7) and (2.10) is our starting point. This system has many important properties.

The numerical results reveal [22, 23, 24, 25, 26, 27, 28, 29]:

(P1) All eigenstates have only positive mass spectra.

(P2) For normal eigenstate, its mean diameter is about 10^3 Compton wave length, and its total energy is almost equal to μc^2 .

(P3) The energy contributed by the nonlinear term and its own electromagnetic field is less than $1eV$ and very stable.

(P4) The abnormal magneton exists for nonlinear Dirac's equation with its own electromagnetic field.

(P5) The different spinor can not take the same eigenstate.

In what follows, we define some classical concepts such as coordinate, speed, mass etc. for an electron ϕ_k respectively, then derive their dynamic equation

Definition 1. *The coordinate $X(t)_k$ and speed v_k of k -th electron e_k are defined respectively by*

$$\vec{X}_k(t) \equiv \int_{R^3} \vec{x} |\phi_k|^2 d^3x = \int_{R^3} \vec{x} q_k^0 d^3x, \quad \vec{v}_k \equiv \frac{d}{dt} \vec{X}_k. \quad (2.11)$$

Definition 2. *Define the 4-dimensional momentum p_k^μ and energy K_k of e_k respectively by*

$$\begin{cases} p_k^\mu \equiv \int_{R^3} \phi_k^\dagger [\hbar i \partial^\mu - e A^\mu + w(Q^\mu - q_k^\mu)] \phi_k d^3x. \\ K_k \equiv p_k^0 + W_k, \quad W_k \equiv \frac{1}{2} w \int \tilde{\gamma}_k^2 d^3x. \end{cases} \quad (2.12)$$

Lemma 1. *By (2.11) and the current conservation law $\partial_\mu q_k^\mu = 0$, we have the speed of e_k*

$$\vec{v}_k = \int \vec{x} \partial_0 q_k^0 d^3x = - \int \vec{x} (\nabla \cdot \vec{q}) d^3x = \int \vec{q}_k d^3x. \quad (2.13)$$

Lemma 2. For any Hermitian operator \hat{P} , and corresponding classical quantity P_k for e_k by

$$P_k \equiv \int_{R^3} \phi_k^+ \hat{P} \phi_k d^3x, \quad (2.14)$$

then we have

$$\frac{d}{dt} P_k = \int_{R^3} \phi_k^+ \left(\partial_t \hat{P} + \frac{i}{\hbar} [\hat{H}_k, \hat{P}] \right) \phi_k d^3x, \quad (2.15)$$

where $[\hat{H}_k, \hat{P}] = \hat{H}_k \hat{P} - \hat{P} \hat{H}_k$.

Theorem 1. For 4-d momentum p_k^μ of each e_k , we have the following rigorous dynamic equation

$$\begin{cases} \frac{d}{dt} p_k^0 = \int \vec{q}_k \cdot (e\vec{E} - w\vec{\mathcal{E}}_k) d^3x - \frac{d}{dt} W_k, \\ \frac{d}{dt} \vec{p}_k = \int [q_k^0 (e\vec{E} - w\vec{\mathcal{E}}_k) + \vec{q}_k \times (e\vec{B} - w\vec{\mathcal{B}}_k)] d^3x, \end{cases} \quad (2.16)$$

where \vec{E} and \vec{B} are electrical and magnetic intensities given in (2.10), $\vec{\mathcal{E}}_k$ and $\vec{\mathcal{B}}_k$ are intensities caused by current except e_k its own current

$$\begin{cases} \vec{\mathcal{E}}_k = -(\nabla Q^0 + \partial_0 \vec{Q}) + (\nabla q_k^0 + \partial_0 \vec{q}_k) = -\sum_{l \neq k} (\nabla q_l^0 + \partial_0 \vec{q}_l), \\ \vec{\mathcal{B}}_k = \nabla \times \vec{Q} - \nabla \times \vec{q}_k = \sum_{l \neq k} \nabla \times \vec{q}_l. \end{cases} \quad (2.17)$$

Proof. For (2.12), we have momentum and energy operator of e_k

$$\begin{cases} \hat{p}_k^0 = \hat{H}_k - eA^0 + w(Q^0 - q_k^0) = \vec{\alpha} \cdot \hat{p}_k + \mu\gamma - \check{\gamma}_k\gamma, \\ \hat{p}_k = -\hbar i \nabla - e\vec{A} + w(\vec{Q} - \vec{q}_k), \quad \hat{K}_k = \hat{p}_k^0 + \frac{1}{2} w \check{\gamma}_k \gamma. \end{cases} \quad (2.18)$$

Then by straightforward calculation we have

$$\begin{cases} \partial_t \hat{K}_k = \vec{\alpha} \cdot \partial_0 [-e\vec{A} + w(\vec{Q} - \vec{q}_k)] - \frac{1}{2} w \partial_0 \check{\gamma}_k \gamma, \\ \partial_t \hat{p}_k = \partial_0 [-e\vec{A} + w(\vec{Q} - \vec{q}_k)], \end{cases} \quad (2.19)$$

$$[\hat{H}_k, \hat{K}_k] = \vec{\alpha} \cdot (-\hbar i \nabla) [-eA^0 + w(Q^0 - q_k^0)] + \frac{1}{2} w (\hat{H}_k \check{\gamma}_k \gamma - \check{\gamma}_k \gamma \hat{H}_k), \quad (2.20)$$

$$\begin{aligned} [\hat{H}_k, \hat{p}_k] &= [eA_0 - w(Q_0 - q_{k0}) + (\mu - w\check{\gamma}_k)\gamma + \vec{\alpha} \cdot \hat{p}_k, \hat{p}_k] \\ &= \hbar i \nabla [eA_0 - w(Q_0 - q_{k0})] + w \hbar i \nabla \check{\gamma}_k \gamma - \vec{\alpha} \times (\hat{p}_k \times \hat{p}_k) \\ &= \hbar i \nabla [eA_0 - w(Q_0 - q_{k0})] + w \hbar i \nabla \check{\gamma}_k \gamma - \\ &\quad \hbar i \vec{\alpha} \times \{ \nabla \times [e\vec{A} - w(\vec{Q} - \vec{q}_k)] \}. \end{aligned} \quad (2.21)$$

So we can derive

$$\begin{aligned}
\frac{d}{dt}K_k &= \int \phi_k^+ [\partial_0 \hat{K}_k + \frac{i}{\hbar} (\hat{H}_k \hat{K}_k - \hat{K}_k \hat{H}_k)] \phi_k d^3x \\
&= \int \phi_k^+ \vec{\alpha} \cdot (\partial_0 [-e\vec{A} + w(\vec{Q} - \vec{q}_k)] + \nabla [-eA^0 + w(Q^0 - q_k^0)]) \phi_k d^3x + \\
&\quad \frac{1}{2} w \int \phi_k^+ \left(-\partial_0 \check{\gamma}_k \gamma + \frac{i}{\hbar} (\hat{H}_k \check{\gamma}_k \gamma - \check{\gamma}_k \gamma \hat{H}_k) \right) \phi_k d^3x \\
&= \int \left(\phi_k^+ \vec{\alpha} \cdot (e\vec{E} - w\vec{\mathcal{E}}_k) \phi_k - \frac{1}{4} w \partial_0 \check{\gamma}_k^2 + \frac{i}{2\hbar} w \phi_k^+ (\hat{H}_k \check{\gamma}_k \gamma - \check{\gamma}_k \gamma \hat{H}_k) \phi_k \right) d^3x \\
&= \int \vec{q}_k \cdot (e\vec{E} - w\vec{\mathcal{E}}_k) d^3x - \frac{1}{4} w \int \partial_0 \check{\gamma}_k^2 d^3x + \\
&\quad \frac{i}{2\hbar} w \int [(\hat{H}_k \phi_k)^+ \check{\gamma}_k \gamma \phi_k - \phi_k^+ \check{\gamma}_k \gamma (\hat{H}_k \phi_k)] d^3x \\
&= \int \left(\vec{q}_k \cdot (e\vec{E} - w\vec{\mathcal{E}}_k) - \frac{1}{4} w \partial_0 \check{\gamma}_k^2 + \frac{1}{2} w \check{\gamma}_k [(\partial_0 \phi_k)^+ \gamma \phi_k + \phi_k^+ \gamma (\partial_0 \phi_k)] \right) d^3x \\
&= \int \vec{q}_k \cdot (e\vec{E} - w\vec{\mathcal{E}}_k) d^3x.
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\frac{d}{dt}\vec{p}_k &= \int \phi_k^+ [\partial_0 \hat{p}_k + \frac{i}{\hbar} (\hat{H}_k \hat{p}_k - \hat{p}_k \hat{H}_k)] \phi_k d^3x \\
&= \int q_k^0 \left(\partial_0 [-e\vec{A} + w(\vec{Q} - \vec{q}_k)] - \nabla [eA_0 - w(Q_0 - q_{k0})] \right) d^3x - \\
&\quad w \int \nabla \check{\gamma}_k^2 d^3x + \int \vec{q}_k \times \left(\nabla \times [e\vec{A} - w(\vec{Q} - \vec{q}_k)] \right) d^3x \\
&= \int [q_k^0 (e\vec{E} - w\vec{\mathcal{E}}_k) + \vec{q}_k \times (e\vec{B} - w\vec{\mathcal{B}}_k)] d^3x
\end{aligned} \tag{2.23}$$

The proof is finished by $\hat{p}_k^0 = K_k - W_k$.

Form the calculation of (2.22) and (2.23) we learn that the forces caused by vector current have the same form as that caused by electromagnetic field. Considering that the corresponding force is much less than that of electromagnetic field, we omit $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ in the following analysis.

3 The Classical Approximation

Assume the external potential is adequately small and the distances among electrons are adequately long, then all $\phi_k(t)$ are solitary waves with central coordinate $\vec{X}_k(t)$. By (2.9), (2.11) and (2.13), we get

$$q_{k0} \rightarrow \delta(\vec{x} - \vec{X}_k), \quad \vec{q}_k \rightarrow \delta(\vec{x} - \vec{X}_k) \vec{v}_k. \tag{3.1}$$

$$W_k = \sqrt{1 - v_k^2} W, \quad W \equiv \frac{1}{2} w \int \check{\gamma}_k^2 d^3 \bar{x}_k, \tag{3.2}$$

where \bar{x}_k stands for the central reference coordinate system of e_k , and W is the proper energy corresponding to nonlinear term. Substituting (3.1) into (2.16), we get the

Newton's second law for all electrons as follows

$$\begin{cases} \frac{d}{dt}p_k^0 = e\vec{v}_k \cdot \vec{E}(\vec{X}_k) - \frac{d}{dt}W_k, \\ \frac{d}{dt}\vec{p}_k = e(\vec{E} + \vec{v}_k \times \vec{B}). \end{cases} \quad (3.3)$$

The proper time of e_k reads $d\tau_k \equiv \sqrt{1 - v_k^2}dt$, the 4 - d speed $u_k^\mu = (1, \vec{v}_k)/\sqrt{1 - v_k^2}$. Correspondingly, the potential equation becomes

$$\partial_\nu \partial^\nu A^\mu = e \sum_{k=1}^N v_k^\mu \delta(\vec{x} - \vec{X}_k) + \rho^\mu, \quad v_k^\mu \equiv (1, \vec{v}_k). \quad (3.4)$$

In what follows, we look for the relation between p_k^μ and u_k^μ , and the definition of inertial mass m_e . By (3.3) we get

$$u_k^\mu \frac{d}{dt}p_{k\mu} = -u_k^0 \frac{d}{dt}W_k = -\frac{W}{\sqrt{1 - v_k^2}} \frac{d}{dt}\sqrt{1 - v_k^2}. \quad (3.5)$$

By (2.3) and the Pauli-Fierz identity, we rewrite (2.6) as

$$[-\hbar i \nabla + e\vec{A} - w(\vec{Q} - \vec{q}_k)]\phi_k = \vec{\alpha}[\hbar i \partial_0 - eA_0 + w(Q_0 - q_{k0})]\phi_k + \vec{S}\phi_k,$$

where $\vec{S} \equiv (S_1, S_2, S_3)$ are all anti-Hermitian matrices. By the above equation and (2.12), we have

$$p_k^\mu = \frac{1}{2} \int_{R^3} [(\hbar i \partial_0 \phi_k)^\dagger \alpha^\mu \phi_k + \phi_k^\dagger \alpha^\mu (\hbar i \partial_0 \phi_k)] d^3x - \int_{R^3} [eA_0 - w(Q_0 - q_{k0})] q_k^\mu d^3x.$$

If e_k moves in a infinitesimal speed \vec{v}_k , then $\hbar i \partial_0 \phi_k \rightarrow \bar{m} \phi_k$, where $\bar{m} > 0$ is a constant. Hence by (3.1) we get

$$p_k^\mu = \int [\bar{m} - eA_0 + w(Q_0 - q_{k0})] q_k^\mu d^3x \equiv m_k u_k^\mu, \quad (3.6)$$

where $m_k = \sqrt{p_k^\mu p_{k\mu}}$ is the scalar mass of e_k . m_k times (3.5) we have

$$\frac{1}{2} \frac{d}{dt}(p_k^\mu p_{k\mu}) = -m_k W \frac{d}{dt} \ln \sqrt{1 - v_k^2}, \quad \text{or} \quad \frac{d}{dt}m_k = -W \frac{d}{dt} \ln \sqrt{1 - v_k^2},$$

so we can define the inertial mass of electron e_k as

$$m_k = m_e + W \ln \frac{1}{\sqrt{1 - v_k^2}}, \quad (3.7)$$

where m_e is static mass of an electron. Substituting (3.7) into (3.6) and (2.12) we get

$$\begin{cases} p_k^\mu = \left(m_e + W \ln \frac{1}{\sqrt{1 - v_k^2}} \right) u_k^\mu \\ K_k = \frac{m_e}{\sqrt{1 - v_k^2}} + W \left(\frac{1}{\sqrt{1 - v_k^2}} \ln \frac{1}{\sqrt{1 - v_k^2}} + \sqrt{1 - v_k^2} \right) \end{cases} \quad (3.8)$$

where m_k is the moving inertial mass of e_k . From (3.8) we find that the nonlinear term violates mass-energy relation slightly. Although (3.8) is derived under the assumption of infinitesimal speed, it is also suitable for the case of high speed, because of the covariant form of p_k^μ .

Since $W \ll m_e$, we omit it from (3.8). Then the Lagrangian corresponding to the coupling system (3.3) and (3.4) reads

$$\mathcal{L} = \sum_{k=1}^N (-m_e - eA_\mu u_k^\mu) \sqrt{1 - v_k^2} \delta(\vec{x} - \vec{X}_k) + eA_\mu \rho^\mu - \frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu. \quad (3.9)$$

If all $|\vec{v}_k| \ll 1$ and the effect of the retarded potential can be ignored, then the nonrelativistic approximation of the total potential becomes

$$A^\mu = V^\mu(\vec{x}) + \frac{e}{4\pi} \sum_{k=1}^N \frac{v_k^\mu}{|\vec{x} - \vec{X}_k|}. \quad (3.10)$$

Omitting the self-potential in (3.9) we get an ordinary differential system

$$\begin{aligned} \mathcal{L}_e &= - \sum_{k=1}^N \left(m_e \sqrt{1 - v_k^2} + eV_\mu v_k^\mu + \frac{e^2}{8\pi} \sum_{l \neq k} \frac{v_{l\mu} v_k^\mu}{|\vec{x} - \vec{X}_l|} \right) \delta(\vec{x} - \vec{X}_k), \\ L &= \int_{R^3} \mathcal{L}_e d^3x = - \sum_{k=1}^N \left(m_e \sqrt{1 - v_k^2} + eV_\mu(\vec{X}_k) v_k^\mu + \frac{e^2}{8\pi} \sum_{l \neq k} \frac{v_{l\mu} v_k^\mu}{|\vec{X}_k - \vec{X}_l|} \right). \end{aligned} \quad (3.11)$$

The Hamiltonian of the system reads

$$H = \sum_{k=1}^N \frac{\partial L}{\partial \vec{v}_k} \cdot \vec{v}_k - L = \sum_{k=1}^N \left(\frac{m_e}{\sqrt{1 - v_k^2}} + eV_{k0} + \frac{e^2}{8\pi} \sum_{l \neq k} \frac{1}{|\vec{X}_k - \vec{X}_l|} \right). \quad (3.12)$$

(3.9), (3.11) and (3.12) constitute the complete theory of the classical mechanics for N -electron.

4 The Nonrelativistic Approximation

Now we make the nonrelativistic approximation and derive the Schrödinger equation for N -electron which move slowly in strong external potential. The conventional quantum mechanics is a linear theory, which defaults the following hypotheses:

Hq.1 The effects of retarded potential can be ignored.

Hq.2 All nonlinear coupling terms can be ignored.

Hq.3 All self coupling potentials, including electromagnetic field caused by electron its own, can be merged into physical mass of electron.

In what follows, we also accept **Hq.1-3** as auxiliary assumptions adding upon equation (2.6) and (2.10). The **Hq.1** means that, only low speeds ($v_k \ll 1$) are considered and the distances among electrons are adequately small. The **Hq.2** means that, the potential caused by electrons is much less than the external potential ($|A_k^\mu| \ll |V^\mu|$), so that the second order terms can be omitted. The **Hq.3** means that the total effect of self-coupling potentials is very small ($\ll \mu$) and stable, so it can be merged into the physical mass of an electron m . The following procedure was once used to derive the Breit potential among electrons with $O(\frac{1}{m^2})$ terms[30]. Here we only keep $O(\frac{1}{m})$ terms to demonstrate how (2.6) implies the N -body Shrödinger equation.

Denote A_k^μ is potential caused by e_k , i.e., $\partial_\alpha \partial^\alpha A_k^\mu = eq_k^\mu$. By **Hq.1** we have solution without retarded potential as follow

$$A_k^\mu = \frac{e}{4\pi} \int_{R^3} \frac{q_k^\mu(t, \vec{X}_k)}{|\vec{x} - \vec{X}_k|} d^3 X_k. \quad (4.1)$$

By **Hq.2** and **Hq.3**, (2.6) becomes

$$(\hbar i \partial_0 - e \Phi_{k0}) \phi_k = \vec{\alpha} \cdot (-\hbar i \nabla + e \vec{\Phi}_k) \phi_k + m \gamma \phi_k, \quad (4.2)$$

where $\Phi_k^\mu = A^\mu - A_k^\mu$ is the external potential with respect to e_k .

Taking conventional transformation

$$\phi_k = \exp\left(\frac{mc^2 t}{\hbar i}\right) \begin{pmatrix} \psi_k \\ \delta_k \end{pmatrix}. \quad (4.3)$$

where ψ_k and δ_k are slowly varying functions of t . Substituting (4.3) into (4.2) and omitting $O(\frac{1}{m^2})$ we get

$$\delta_k = \frac{1}{2m} \vec{\sigma} \cdot (-\hbar i \nabla + e \vec{\Phi}_k) \psi_k, \quad (4.4)$$

$$\hbar i \partial_0 \psi_k = e \Phi_{k0} \psi_k + \vec{\sigma} \cdot (-\hbar i \nabla + e \vec{\Phi}_k) \delta_k. \quad (4.5)$$

Substituting (4.4) into (4.5) we get the Pauli's equation for each e_k

$$\hbar i \partial_t \psi_k = \left(-\frac{\hbar^2}{2m} \Delta + e \Phi_{k0} + \mu_z \vec{\sigma} \cdot \vec{B} + \frac{e}{m} \vec{\Phi}_k \cdot (-\hbar i \nabla) + \frac{e^2}{2m} \vec{\Phi}_k^2 \right) \psi_k. \quad (4.6)$$

where $\mu_z = \frac{|e|\hbar}{2m}$ is the Bohr magneton. The Coulomb gauge $\nabla \cdot \vec{\Phi}_k = 0$ holds in (4.6) by **Hq.1**.

Since the magnetic field caused by \vec{A}_k is much smaller than that of external potential, we also omit it for simplicity. Let x_3 be parallel with \vec{B} , then

$$\vec{\sigma}_k \cdot \vec{B} = \sigma_3 B = \lambda_k B, \quad \lambda_k = 1 \text{ or } -1.$$

Hence one component of ψ_k vanishes, so all ψ_k can be treated as a scalar. By **Hq.2**, (4.6) becomes

$$\hbar i \partial_t \psi_k = \left[\frac{1}{2m} \hat{P}^2 + eV_0 + \lambda_k \mu_z B + e \sum_{l \neq k} \left(A_{l0} + \frac{1}{m} \vec{A}_l \cdot \hat{P} \right) \right] \psi_k, \quad (4.7)$$

where $\hat{P} = -\hbar i \nabla + e\vec{V}$. Substituting (4.4) into (2.5) we get

$$q_{k0} = |\psi_k|^2, \quad \vec{q}_k = \frac{1}{2m} [\psi_k^+ \hat{P} \psi_k + (\hat{P} \psi_k)^+ \psi_k]. \quad (4.8)$$

By (4.1) and **Hq.1** we get

$$A_{k0} = \frac{e}{4\pi} \int_{R^3} \frac{|\psi_k(t, \vec{X}_k)|^2}{|\vec{x} - \vec{X}_k|} d^3 X_k, \quad \vec{A}_k = \frac{e}{4\pi} \int_{R^3} \frac{\psi_k^+ \hat{P} \psi_k}{m |\vec{x} - \vec{X}_k|} d^3 X_k, \quad (4.9)$$

where

$$\hat{P}_k = -\hbar i \nabla_k + \vec{V}(t, \vec{X}_k) = -\hbar i \frac{\partial}{\partial \vec{X}_k} + \vec{V}(t, \vec{X}_k).$$

Substituting (4.9) into (4.7) we get the separating form for nonrelativistic quantum mechanics for N -electron as follows

$$\hbar i \partial_t \psi_k = \left(\frac{1}{2m} \hat{P}^2 + eV_0 + \lambda_k \mu_z B + \frac{e^2}{4\pi} \sum_{l \neq k} \int_{R^3} \frac{|\psi_l(t, \vec{X}_l)|^2}{|\vec{x} - \vec{X}_l|} d^3 X_l \right) \psi_k, \quad (4.10)$$

The action corresponds to (4.10) is given by

$$\begin{aligned} \mathbf{I} &= \sum_{k=1}^n \int_{t_0}^t dt \int \psi_k^+(t, \vec{x}) (\hbar i \partial_t - \hat{\mathbf{H}}_k) \psi_k(t, \vec{x}) d^3 x \\ &= \sum_{k=1}^n \int_{t_0}^t dt \int \psi_k^+(t, \vec{X}_k) (\hbar i \partial_t - \hat{\mathbf{H}}_k) \psi_k(t, \vec{X}_k) d^3 X_k, \end{aligned} \quad (4.11)$$

where all coordinates are integral variables, i.e., they are dummy arguments, so we can assign \vec{X}_k to the electron e_k ,

$$\hat{\mathbf{H}}_k = \frac{1}{2m} \hat{P}_k^2 + eV_0(t, \vec{X}_k) + \lambda_k \mu_z B(t, \vec{X}_k) + \frac{e^2}{8\pi} \sum_{l \neq k} \int_{R^3} \frac{|\psi_l(t, \vec{X}_l)|^2}{|\vec{X}_k - \vec{X}_l|} d^3 X_l. \quad (4.12)$$

Noticing the normalizing conditions

$$1 = \int_{R^3} |\psi_k(t, \vec{X}_k)|^2 d^3 X_k, \quad (k = 1, \dots, N), \quad (4.13)$$

Multiplying (4.11) by (4.13), we get a combined form of action

$$\mathbf{I} = \int_{t_0}^t dt \int_{R^{3n}} \Psi^+ (\hbar i \partial_t - \hat{\mathbf{H}}) \Psi d^3 X_1 d^3 X_2 \cdots d^3 X_n, \quad (4.14)$$

where

$$\Psi(t, \vec{X}_1, \vec{X}_2, \cdots, \vec{X}_n) = \psi_1(t, \vec{X}_1) \psi_2(t, \vec{X}_2) \cdots \psi_n(t, \vec{X}_n), \quad (4.15)$$

the total Hamiltonian operator is given by

$$\hat{\mathbf{H}} = \sum_{k=1}^n \left(\frac{1}{2m} \hat{P}_k^2 + eV_0(t, \vec{X}_k) + \lambda_k \mu_z B(t, \vec{X}_k) + \frac{e^2}{8\pi} \sum_{l \neq k} \frac{1}{|\vec{X}_l - \vec{X}_k|} \right). \quad (4.16)$$

By variation of (4.14) with respect to Ψ , we finally get the standard Schrödinger equation for N -electron

$$\hbar i \partial_t \Psi = \hat{\mathbf{H}} \Psi. \quad (4.17)$$

Comparing (4.16) with (3.12), we find that the Hamiltonian of the classical mechanics and quantum mechanics have just the same structure, except a magneton term which is ignored in classical approximation.

5 Discussion and Conclusion

The above derivation clearly shows how a self consistent system (2.6) naturally implies the classical and quantum mechanics for many-body. But the nonlinear coupling potential in (2.4) is not finally determined. What is more important for this work is that, it shows us there are different solving methods for a specific physical problem.

In the procedure of transition from the classical mechanics to the quantum theory, the correspondence principle is used, i.e., the classical quantities are translated into the corresponding operators in quantum mechanics by analogy. From the above derivation we find that the analogies are fortunately valid for some cases. The Schrödinger equation is a typical example of success. But analogy is not reliable generally, we should keep some suspicion and caution in mind while doing so, rather than treat it absolutely.

The procedure of second quantization seems to expand one Dirac's equation into many ones by applying Fourier transform and the principle of superposition. It seems to be closely related with the dynamic equation (2.6). However this procedure is mainly effective for linear equation, so it must continuously introduce extra conditions

such as the principle of exclusion, positive energy condition, quantum condition, box normalizing condition, renormalizing procedure, etc. as compensations to keep the quantum theory consistent.

Despite each approach to the many body quantum mechanics starting from the different points, it is surprising that all approaches have achieved successes to some degree for several cases. There should be some intrinsic relations between these theories, in which the action principle seems to play a central role and act as a connecting bridge.

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